

Irreducibility of moduli of curves

- Clebsch-Hurwitz argument ($\setminus \mathbb{Q}$)
- Algebraic argument by theory of admissible covers ($\text{char} = 0$)
- Arguments for $\text{char } p > 0$ (Deligne-Mumford, Fulton)

Goal. $\overline{\mathcal{M}}_{g,n}$ is irreducible

Remark. (1) $\overline{\mathcal{M}}_{g,n}$ irreducible $\Leftrightarrow \overline{\mathcal{M}}_{g,n}$ connected
 $\Leftrightarrow \overline{\mathcal{M}}_g$ connected ($\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ is universal)
 $\Leftrightarrow \mathcal{M}_g$ connected & dense in $\overline{\mathcal{M}}_g$

(2) $\overline{\mathcal{M}}_{g,n} \xrightarrow{\text{cms}} \overline{\mathcal{M}}_{g,n}$ satisfy $|\overline{\mathcal{M}}_{g,n}| = |\overline{\mathcal{M}}_{g,n}|$
 $\Rightarrow$ statements in (1) are equivalent to some statements on coarse moduli spaces

Consequences

- (1) $\mathcal{M}_g$ connected $\Rightarrow$ genus is the only discrete invariant
- (2) $\mathcal{M}_g \subsetneq \overline{\mathcal{M}}_g \Rightarrow \nexists$ component of $\overline{\mathcal{M}}_g$ that consists of singular curves

Branched Coverings

Def. A branched covering $f: C \rightarrow D$ is a finite separable morphism of smooth connected curves.

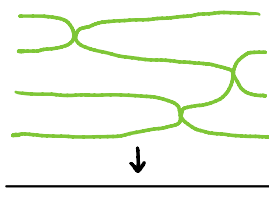
- If f tamely ramified at $P \in C(k)$, then
 $\text{length}(\Omega_{C/D})_P = e_P - 1$ (e_P : ramification index)
- $0 \rightarrow f^* \Omega_D \rightarrow \Omega_C \rightarrow \Omega_{C/D} \rightarrow 0$
 $\Rightarrow K_C = f^* K_D + R$ ($R = \sum_{P \in C} (\Omega_{C/D})_P \cdot P$)
 $\Rightarrow 2g(C) - 2 = (\deg f)(2g(D) - 2) + \deg R$ (RH)

Def. A branched covering $f: C \rightarrow D$ is simply branched if

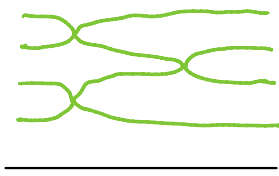
- (1) $\forall$ ramification point $\in C(k)$ has index 2 and is tamely ramified
- (2) there's at most one ramification point in each fiber

By RH, a simply branched covering $C \rightarrow \mathbb{P}^1$ w/ $\deg = d$ is ramified over $b = 2d + 2g - 2$ points in $\mathbb{P}^1$.

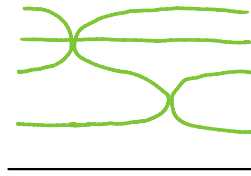
Ex.



simply
branched



$\mathbb{P}^1$



not simply
branched

$\mathbb{P}^1$

Lemma. $C =$ smooth projective curve of genus g / $k = \bar{k}$ char 0.

$L =$ line bundle s.t. $\deg(L) \geq 2g + 3$.

Then $\forall$ general $V \subset H^0(C, L)$ w/ $\dim = 2$,

$C \xrightarrow{|V|} \mathbb{P}^1$ simply branched.

pf. $h^0(L) = d + 1 - g \Rightarrow \dim \operatorname{Gr}(2, H^0(L)) = 2(d - g - 1)$.

If $C \dashrightarrow \mathbb{P}^1$ doesn't satisfy condition, then either

(a) V has a basepoint

(b) $\exists$ ram. pt with index > 2

(c) $\exists 2$ ram. pts in the same fiber

WTS : each locus has $\dim < 2(d - g - 1)$

(a) L is basepoint free.

Fix $p \in C$. $\dim \{s \in H^0(C, L) \mid s \text{ vanishes at } p\} \leq d-g$.

So $\dim(\text{locus of } V \text{ w/ basepoint at } p) \leq 2(d-g-2)$.

$\dim(\text{locus}) \leq 2(d-g-2) + 1 < 2(d-g-1)$.

(b) equiv. to $\exists p \in C, s \in V \setminus \{0\} : s \in H^0(C, L(-3p))$.

$\dim(\text{locus of } V \text{ w/ branch point w/ index } \geq 3 \text{ at } p)$

$$= \dim \mathbb{P} H^0(C, L(-3p)) + \dim \mathbb{P} (H^0(C, L) / \langle s \rangle) = 2d-2g-4.$$

$\overset{d-g-3}{\parallel}$ $\overset{d-g-1}{\parallel}$

$$\Rightarrow \dim(\text{locus}) = (2d-2g-4) + 1 < 2(d-g-1).$$

(c) equiv. to $\exists p \neq q \in C, s \in V \setminus \{0\} : s \in H^0(C, L(-2p-2q))$.

$\dim(\text{locus of } V \text{ w/ two ramified points } p, q)$

$$= \dim \mathbb{P} H^0(C, L(-2p-2q)) + \dim \mathbb{P} (H^0(C, L) / \langle s \rangle) = 2d-2g-5$$

$\overset{d-g-4}{\parallel}$ $\overset{d-g-1}{\parallel}$

$$\Rightarrow \dim(\text{locus}) = (2d-2g-5) + 2 < 2(d-g-1). \quad \square$$

Lemma. $C \rightarrow \mathbb{P}^1$ simply branched covering w/ $\deg \geq 2$. ($\text{char} = 0$)

$$\Rightarrow \text{Aut}(C/\mathbb{P}^1) = \{1\}.$$

pf. Every $\phi \in \text{Aut}(C/\mathbb{P}^1)$ must fix the $2d+2g-2$ branched points.

(Mayer) $\neq$ nontrivial auts fixing $\geq 2g+2$ points. $\square$

Def. $H_{d,b} :=$ moduli space of $C \rightarrow \mathbb{P}^1$ simply branched coverings of deg d over b points

(can be defined as a top space or an alg. space)

(if / C)

↪ since $\neq$ stackiness

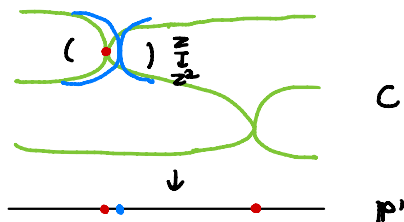
$$H_{d,b} \begin{matrix} \swarrow \\ \mathcal{M}_g \end{matrix} \begin{matrix} \searrow \\ \text{Sym}^b \mathbb{P}^1 \setminus \Delta \cong \mathbb{P}^b \setminus \Delta \end{matrix}$$

$$[C \rightarrow \mathbb{P}^1] \begin{matrix} \swarrow \\ [C] \end{matrix} \begin{matrix} \searrow \\ (b \text{ branched points}) \end{matrix}$$

Lemma. In char 0, $H_{d,b} \rightarrow \text{Sym}^b \mathbb{P}^1 / \Delta$ finite & étale. (assume $d \geq g$)

Pf. finiteness $\Rightarrow$ later!

(topological / C) étale = local homeomorphism



(algebraic)

$$\underline{\text{WTS}} \quad \text{Def}^{\text{lst}}(C \xrightarrow{f} \mathbb{P}^1) \rightarrow \text{Def}^{\text{lst}}(\{p_1, \dots, p_b\} \hookrightarrow \mathbb{P}^1) \quad \text{bij.}$$

$\parallel$
 $H^0(C, N_f)$ $\dim = b$

$$0 \rightarrow T_C \rightarrow f^* T_{\mathbb{P}^1} \rightarrow N_f \rightarrow 0$$

$$0 \rightarrow H^0(C, f^* T_{\mathbb{P}^1}) \rightarrow H^0(C, N_f) \rightarrow H^1(C, T_C) \rightarrow H^1(C, f^* T_{\mathbb{P}^1})$$

$\dim = 2d+1-g$ $\downarrow$
 $\dim = b$ $\dim = 3g-3$ $\overset{0}{\parallel}$
 $\dim = 0$

More topological inputs / C

Riemann Existence Theorem

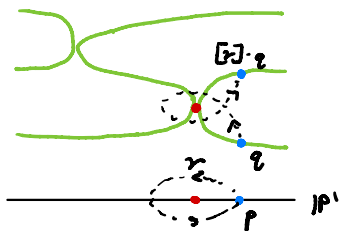
$$\begin{array}{ccc}
 \exists \text{ b.i.s } \{ C \rightarrow \mathbb{P}^1 \text{ algebraic branched coverings} \} & & \\
 \downarrow \uparrow & & \\
 \{ \simeq \text{ topological } \simeq \} & & \\
 \uparrow \downarrow & & \\
 \{ \simeq \text{ holomorphic } \simeq \} & &
 \end{array}
 \quad \text{(hard!)}$$

Monodromy actions

$C \xrightarrow{f} \mathbb{P}^1$ (topological) branch covering, $B \subset \mathbb{P}^1$ ramification locus.

Fix basepoint $p \in \mathbb{P}^1 \setminus B$. ($\deg(f) = d$)

Def. (monodromy action $\pi_1(\mathbb{P}^1 \setminus B, p) \curvearrowright f^{-1}(p)$)



$$\begin{aligned}
 & \forall [\gamma] \in \pi_1(\mathbb{P}^1 \setminus B, p), q \in f^{-1}(p): \\
 & \gamma: [0, 1] \rightarrow \mathbb{P}^1 \setminus B \text{ lifts uniquely to} \\
 & \tilde{\gamma}: [0, 1] \rightarrow C \text{ s.t. } \tilde{\gamma}(0) = q. \\
 & \Rightarrow [\gamma] \cdot q = \tilde{\gamma}(1).
 \end{aligned}$$

Prop. $B = \{p_1, \dots, p_b\} \subset \mathbb{P}^1$ finite set, $p \in \mathbb{P}^1 \setminus B$ fixed point, $b > 0$.

$\exists$ canonical bijection

$$\left\{ C \rightarrow \mathbb{P}^1 \text{ deg } d \text{ branched coverings w/ ram. locus } B \right\} / \sim_{\text{isom}} \xleftrightarrow{(\text{bij.})} \left\{ \begin{array}{l} \text{group homo } \pi_1(\mathbb{P}^1 \setminus B, p) \xrightarrow{f} S_d \\ \text{s.t. } \text{In}(p) \subseteq S_d \text{ transitive subgroup} \end{array} \right\} / \sim$$

$$(p \sim p' \Leftrightarrow \exists h \in S_d: p' = h^{-1} p h)$$

$$\begin{aligned}
 & \cup \\
 & \left\{ \text{simply branched coverings} \right\} / \sim \xleftrightarrow{(\text{bij.})} \left\{ p \text{ s.t. } p(\sigma_1), \dots, p(\sigma_b) \in S_d \right\} / \sim \\
 & \quad \text{are transpositions}
 \end{aligned}$$

(Note) $\pi_1(\mathbb{P}^1 \setminus B, p) = \langle \sigma_i \mid \sigma_1 \dots \sigma_b = 1 \rangle$ (σ_i = simple loops around p_i)

The Clebsch - Hurwitz argument ($\setminus \mathbb{C}$)

Fix $d \geq 2g+3$, and let $b = 2d+2g-2$.

$H_{d,b} \rightarrow \mathcal{M}_g$ is surj. ($\forall C/\mathbb{C}$ of genus g : $\exists$ simply branched cover $C \rightarrow \mathbb{P}^1$)
 $\Rightarrow$ connectedness of $H_{d,b}$ implies connectedness of $\mathcal{M}_g$.

Theorem. (Clebsch - Hurwitz) $H_{d,b}$ is connected.

pf. Recall $\beta: H_{d,b} \rightarrow \text{Sym}^b \mathbb{P}^1 \setminus \Delta$ is a finite covering space.

$\Rightarrow$ suffices to show $\forall B \in \text{Sym}^b \mathbb{P}^1 \setminus \Delta$:

$\pi_1(\text{Sym}^b \mathbb{P}^1 \setminus \Delta, B) \curvearrowright \beta^{-1}(B) =: H_{d,B}$ is transitive.

$H_{d,B} = \{ \text{degree } d \text{ coverings } C \rightarrow \mathbb{P}^1 \text{ simply branched over } B \} / \sim$

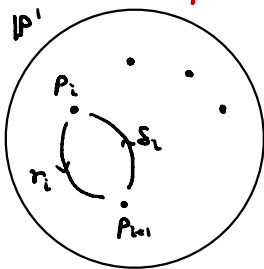
$= \{ \text{group homs } \rho: \pi_1(\mathbb{P}^1 \setminus B, p) \rightarrow S_d$

s.t. $\text{Im}(\rho)$ transitive and $\rho(\sigma_1), \dots, \rho(\sigma_b)$ transpositions $\} / \sim$

$= \{ (\tau_1, \dots, \tau_b) \in (S_d)^b \mid \tau_i \text{ transpositions and } \tau_1 \dots \tau_b = 1 \} / \sim$

Strategy Find enough elts in $\pi_1(\text{Sym}^b \mathbb{P}^1 \setminus \Delta, B)$ that acts on $H_{d,B}$

in a prescribed way.



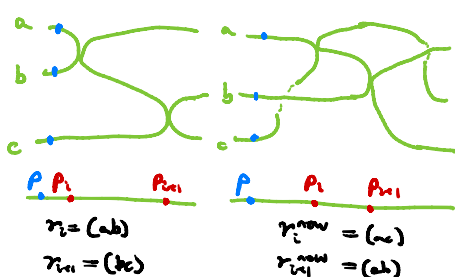
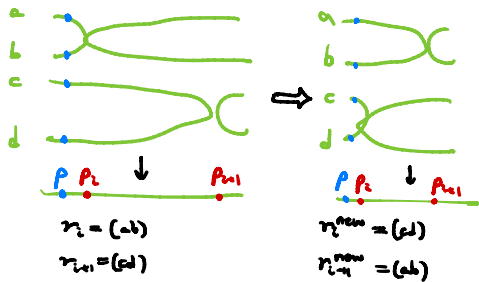
$T_i: [0, 1] \rightarrow \text{Sym}^b \mathbb{P}^1 \setminus \Delta \quad (\forall 1 \leq i \leq b-1)$

$t \mapsto (p_1, \dots, p_{i-1}, \gamma_i(t), \delta_i(t), p_{i+2}, \dots, p_b)$

$\Rightarrow T_i \cdot (\tau_1, \dots, \tau_b) = (\tau_1, \dots, \tau_{i-1}, \tau_i^{-1} \tau_{i+1} \tau_i, \tau_i, \tau_{i+1}, \dots, \tau_b)$

Applying T_i 's, we can move any $(\tau_1, \dots, \tau_b)$

to $\tau^* = ((12), (12), (13), (13), \dots, (1(b-1)), (1(b-1)), \underbrace{(12), \dots, (12)}_{2g+2})$.



Irreducibility using admissible covers (char = 0)

* Prop ("every smooth curve degenerates to a singular stable curve")

C smooth, connected, projective curve of genus g / $k = \bar{k}$ char 0.

$\Rightarrow \exists$ connected curve T w/ $t_1, t_2 \in T$ and a family $\mathcal{C} \rightarrow T$ of stable curves s.t. $\mathcal{C}_{t_1} \cong C$ and $\mathcal{C}_{t_2}$ is a singular stable curve.

Pf Step 1 Construct the prototype (for the family)

- Pick $d \geq 2g+3 \Rightarrow$ simply branched covering $C \rightarrow \mathbb{P}^1$
(branched over $\{p_1, \dots, p_b\}$, $b = 2d + 2g - 2$)
- Defines $G = [(\mathbb{P}^1, p_1, \dots, p_b)] \in \mathcal{M}_{0,b}$.
- $H_{d,b} \rightarrow \text{Sym}^b \mathbb{P}^1$ étale $\Rightarrow$ open $\Rightarrow$ Image = open dense, so we may assume $G \in \mathcal{M}_{0,b}$ general.
- $\overline{\mathcal{M}}_{0,b}$ connected $\Rightarrow G$ degenerates to

$$D_0 = \left[\underbrace{\text{[Diagram of } b-2 \text{ nodes with } \mathbb{P}^1 \text{'s branching]}_{b-2 \text{ } \mathbb{P}^1 \text{'s}}} \dots \right] \in \overline{\mathcal{M}}_{0,b}$$

$\left(\begin{array}{l} \Delta = \text{Spec } k[t]_{(t)} \\ \Delta^* = \text{Spec } k(t) \end{array} \right) \quad \exists \text{ family of } b\text{-pointed stable curves } (\mathcal{D} \rightarrow \Delta, \sigma_i)$
 s.t. $\begin{cases} \text{generic fiber } (\mathcal{D}^*, \sigma_i^*) \cong (\mathbb{P}^1, p_1, \dots, p_b) \times_{\Delta^*} \Delta^* \\ \text{special fiber } (\mathcal{D}_0, \sigma_i(0)) = (D_0, \{q_i\}) \end{cases}$

$$\begin{array}{ccccc} C & \longleftarrow & \mathcal{C}^* & \hookrightarrow & \mathcal{C} \\ \downarrow & & \downarrow & & \vdots \\ \mathbb{P}^1_k & \longleftarrow & \mathcal{D}^* & \hookrightarrow & \mathcal{D} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec } k & \longleftarrow & \Delta^* & \hookrightarrow & \Delta \end{array}$$

$\mathcal{C}$ = normalization of $\mathcal{D}_0$
inside $K(\mathcal{C}^*)$

Step 2 Change $\mathcal{C} \rightarrow \Delta$ appropriately

- by "purty of branch locus,"
the ramification locus of $\mathcal{C} \rightarrow \mathcal{D}$
is a divisor in the relative smooth
locus of $\mathcal{C} \rightarrow \mathcal{D}$

$\Rightarrow \mathcal{C}_0 \rightarrow \mathcal{D}_0$ ramified over

$\{\sigma_i\} = \{\sigma_i(0)\}$ and possibly
over irreducible components of $\mathcal{D}_0$

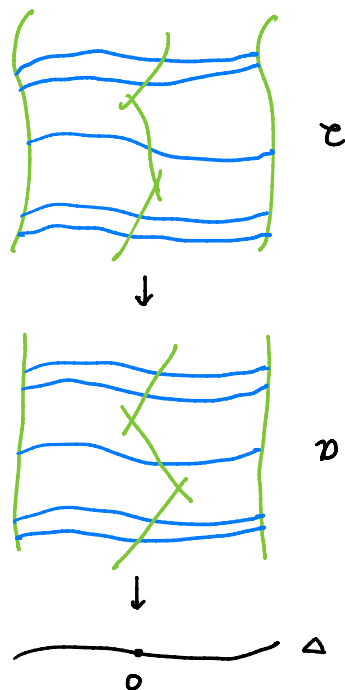
- After suitable base change $\Delta \rightarrow \Delta (\tau \mapsto \tau^m)$
& replacing $\mathcal{C} \mapsto (\mathcal{C} \times_{\Delta} \Delta)^{\text{nor}}$,

$\mathcal{C}_0 \rightarrow \mathcal{D}_0$ is ramified only over
 $\{\sigma_i\}$ and nodes.

- One can check $\mathcal{C}_0$ is nodal, and $\mathcal{C} \rightarrow \Delta$ family of nodal
curves.

- Finally, obtain the stable model $\mathcal{C}^{\text{st}} \rightarrow \Delta$.

Since $\{\sigma_i\} \subseteq \text{Im}(\mathcal{C}_0^{\text{st}} \rightarrow \mathcal{D}_0)$, $\mathcal{C}_0^{\text{st}}$ cannot be
irreducible and hence singular. $\square$



Prop. Assuming $\overline{\mathcal{M}}_{g',n}$ irreducible for $g' < g$,

$\mathcal{S} = \overline{\mathcal{M}}_g / \mathcal{M}_g$ is connected.

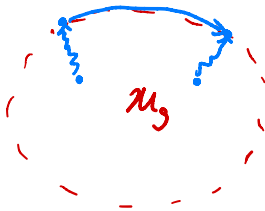
pf Recall $\mathcal{S} = \mathcal{S}_0 \cup \dots \cup \mathcal{S}_{\lfloor \frac{g}{2} \rfloor}$ where $\begin{cases} \mathcal{S}_0 = \text{Im}(\overline{\mathcal{M}}_{g-1,2} \rightarrow \overline{\mathcal{M}}_g) \\ \mathcal{S}_i = \text{Im}(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \rightarrow \overline{\mathcal{M}}_g) \end{cases}$
 By hypothesis, $\mathcal{S}_0, \mathcal{S}_i$ all connected.
 Also, they intersect mutually. ($1 \leq i \leq \lfloor \frac{g}{2} \rfloor$)

$$\left[\begin{array}{c} g-i-j \\ \text{diagram} \\ i \quad j \end{array} \right] \in \mathcal{S}_i \cap \mathcal{S}_j$$

$$\left[\begin{array}{c} g-i-1 \\ \text{diagram} \\ i \end{array} \right] \in \mathcal{S}_0 \cap \mathcal{S}_i$$

Theorem. $\overline{\mathcal{M}}_{g,n}$ is irreducible.

pf



Irreducibility in char $p > 0$

Deligne - Mumford's argument

Fact. $X \rightarrow Y$ smooth & proper.

The function $Y \rightarrow \mathbb{Z}$ ($y \mapsto \#$ connected components of $X_{\overline{y}}$)
is constant.

$\overline{\mathcal{M}}_g \rightarrow \text{Spec } \mathbb{Z}$ smooth and proper.

So $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} \mathbb{C}$ connected $\Rightarrow \overline{\mathcal{M}}_g \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$ connected.

Fulton's argument (works for $p > g+1$)

Hurwitz scheme $H_{d,b}$ defined over $\mathbb{Z}$ & $\exists$ diagram

$$\begin{array}{ccc} & H_{d,b} & \leftarrow \text{finiteness for } p > g+1 \\ \swarrow & & \searrow \text{étale} \\ \mathcal{M}_g & & \text{Sym}^b \mathbb{P}^1 \setminus \Delta \end{array} \quad (\text{same as before})$$

Establishes a "reduction theorem":

since $H_{d,b} \rightarrow \text{Sym}^b \mathbb{P}^1 \setminus \Delta$ finite étale for $p > g+1$,

$H_{d,b} \times_{\mathbb{Z}} \mathbb{C}$ connected $\Rightarrow H_{d,b} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$ connected ($p > g+1$)